PHYS4150 — PLASMA PHYSICS

LECTURE 20 - PLASMA WAVES

Sascha Kempf[*]

G135, University of Colorado, Boulder

Fall 2020

---

1 PLASMA WAVES

In lecture 4 we derived the plasma frequency using the rather limited set of tools available to us at this time. Now we will use the whole armory of methods at our disposal to have an in-depth look at plasma oscillations. The mathematical structure of any (linear) wave is $\sim e^{i(\mathbf{k}\mathbf{x}-\omega t)}$. When thinking about oscillations and waves it is always useful to think about "what is the restoring force". We can group plasma waves based on that into *Electrostatic Waves* and Electromagnetic (EM) Waves.

1.1 *Electrostatic Waves*

Here, an electric field is generated by charge separation and only comes from the *Poisson* equation. The particles must "sloshing" to "bunch up" in $\mathbf{k}$ direction. It is a distinctive property of such waves that they are not associated with magnetic fields.

1.2 *Electromagnetic Waves*

Here, the wave's electric field comes from *Poisson* and *Faraday's law* ($\mathbf{E} = \nabla \times \mathbf{B}$). The wave's magnetic field results from *Ampere's law* $\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + c^{-2}\dot{\mathbf{E}}$. The particle motion can be both, parallel or orthogonal to the wave vector $\mathbf{k}$.

1.3 *General approach*

1. Decide on geometry
2. Decide on equations
3. Linearize equations
4. Eliminate fluctuating variables to get the dispersion relation $\omega(\mathbf{k})$

[*]sascha.kempf@colorado.edu

"Linearize" means to assume that the waves create a small perturbation (usually denoted by δ).

$$n(\mathbf{x},t) = n_0 + \delta n\, e^{i(\mathbf{kx}-\omega t)} \qquad \delta n \ll n_0$$

$$\mathbf{E}(\mathbf{x},t) = \mathbf{E}_0 + \delta\mathbf{E}\, e^{i(\mathbf{kx}-\omega t)} \qquad \mathbf{E}_0 = 0$$

$$\mathbf{v}(\mathbf{x},t) = \mathbf{v}_0 + \delta\mathbf{v}\, e^{i(\mathbf{kx}-\omega t)} \qquad \mathbf{v}_0 = 0$$

$$p(\mathbf{x},t) = p_0 + \delta p\, e^{i(\mathbf{kx}-\omega t)}$$

Higher order terms are generally neglected. Notice that in this formulation, perturbations are made up of spatial and temporal Fourier modes. Our job is to find out which ω's and $\mathbf{k}$'s can exist, and whether certain ω's only exist with certain $\mathbf{k}$'s. Notice further that

$$\frac{\partial}{\partial \mathbf{x}} \equiv i\mathbf{k}$$

and

$$\frac{\partial}{\partial t} \equiv -i\omega,$$

i.e. we can replace the temporal and spatial derivatives by the expressions given above.

2 ELECTROSTATIC WAVES

There are two kinds of such waves. High frequency waves oscillate that fast that ions "don't have time" to move, while low frequency waves oscillate slow enough for the ions to follow.

2.1 *Plasma Oscillations*

So, what does actually happen if we take away some electrons (say 1%) from some region? Let's employ the general approach we just discussed:

Geometry

$$\mathbf{k} = (k,0,0)$$

$$\mathbf{E}_0 = 0 \qquad \delta\mathbf{E} = (\delta E,0,0)$$

$$\mathbf{B}_0 = 0 \qquad \delta\mathbf{B} = (0,0,0) \qquad \text{(electrostatic)}$$

$$\mathbf{v}_0 = 0 \qquad \delta v \neq 0 \qquad \text{(no flow)}$$

$$n_0 \neq 0$$

Poisson equation:

$$
\begin{aligned}
\frac{\partial E}{\partial x} &= \frac{e}{\epsilon_0}(n_i - n_e) \\
\frac{\partial E_0}{\partial x} + \frac{\partial \delta E}{\partial x} &= \frac{e}{\epsilon_0}(n_{0i} + \delta n_{0i} - n_{0e} - \delta n_{0e}) \\
\frac{\partial \delta E}{\partial x} &= -\frac{e}{\epsilon_0}\delta n \\
ik\delta E &= -\frac{e}{\epsilon_0}\delta n \\
\delta E &= \frac{i e}{\epsilon_0 k}\delta n
\end{aligned}
$$

Continuity equation:

$$
\begin{aligned}
\frac{\partial n_e}{\partial t} + \nabla(n_e v_e) &= 0 \\
\frac{\partial n_{e0}}{\partial t} + \frac{\partial \delta n_e}{\partial t} + \frac{\partial}{\partial x}\left[(n_{e0} + \delta n_{e0})(v_{e0} + \delta v_{e0}))\right] &= 0 \\
\frac{\partial n_{e0}}{\partial t} + \frac{\partial \delta n_e}{\partial t} + \frac{\partial}{\partial x}\left[n_{e0}v_{e0} + n_{e0}\delta v_{e0} + \delta n_{e0}v_{e0} + \delta n_{e0}\delta v_{e0}\right] &= 0 \\
-i\omega\delta n + ikn_0\delta v &= 0 \\
\delta v &= \frac{\omega}{k}\frac{\delta n}{n_0}
\end{aligned}
$$

Momentum equation

$$
\begin{aligned}
m_e\frac{\partial v}{\partial t} &= -eE \\
m_e\frac{\partial v_0}{\partial t} + m_e\frac{\partial \delta v}{\partial t} &= -eE_0 - e\delta E \\
-i\omega\delta v &= -\frac{e}{m_e}\delta E \\
\delta E &= \frac{i\omega\delta v m_e}{e}
\end{aligned}
$$

and hence

$$
\begin{aligned}
\frac{i e}{\epsilon_0 k}\delta n &= \frac{i\omega m_e}{e}\delta v \\
\frac{i e}{\epsilon_0 k}\delta n &= \frac{i\omega m_e}{e}\frac{\omega}{k}\frac{1}{n_0}\delta n
\end{aligned}
$$

The resulting frequency

$$\boxed{\omega^2 = \frac{n_0 e^2}{\epsilon_0 m_e}}$$

is the *plasma frequency*, which we already derived in lecture 4 using “handwaving” arguments.